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Nan Wang

Actuarial Research Paper No. 158

July 2004

ISBN 1 901615 80 4

Cass Business School
106 Bunhill Row
London EC1Y 8TZ
T +44 (0)20 7040 8470
www.cass.city.ac.uk

An asset allocation strategy for a risk reserve considering both risk and profit

Nan Wang*

Faculty of Actuarial Science and Statistics

Cass Business School, City University, London

Abstract Consider the risk reserve of an insurer

$$R_t = U + ct - \sum_{i=1}^{N_t} Y_i, \;\; t \geq 0$$

where U is the initial reserve, c is the premium income rate and $\sum_{i=1}^{N_t} Y_i$ is the claim process. With a utility-based approach, we show that there are investment strategies which will chang the above reserve process into

$$R_t = \rho B_t + U + (c + c_0)t - \sum_{i=1}^{N_t} Y_i,$$

which is almost the same as Gerber's extension of the classical risk model (Gerber (1970)). Here B_t is the Brownian motion underlying the dynamics of the stock index (Black-Scholes model), ρ and c_0 are positive and related to the market return, market volatility and the utility choice. Properly selected utilities will make this process both safer and more profitable than the original process without investment.

Keywords. Risk reserve, optimal investment strategy, martingale approach, ruin probability, adjustment coeficient.

1. Introduction

Consider the risk reserve process (collective risk model) of an insurer

$$R_t = U + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0 \tag{1}$$

where U is the initial reserve of an insurer, c is the premium income rate, N_t is the number of claims by time point t and $Y_i, i \geq 1$ are the amounts of successive claims.

E-mail address: n.a.n.wang@city.ac.uk (N. Wang).

The claim process $\{\sum_{i=1}^{N_t} Y_i, t \geq 0\}$ is assumed to have independent increments and, for any finite t,

$$E\left(\sum_{i=1}^{N_t} Y_i\right)^{2^+} < \infty. \tag{2}$$

Here, by $E(\cdot)^{2^+} < \infty$, we imply that there exists a positive $\epsilon > 0$ (which could be related to the item inside the bracket) such that $E(\cdot)^{2+\epsilon} < \infty$.

In this paper, we consider whether the insurer may get a better situation by investing the risk reserve with a proper investment strategy. We adopt the basic and standard Black-Scholes market model, where there are two types of assets, a riskless asset (bond or money market account) and a risky asset (stock index) with price dynamics

$$dX_t^{(0)} = rX_t^{(0)}dt, \quad dX_t^{(1)} = \mu X_t^{(1)}dt + \sigma X_t^{(1)}dB_t \tag{3}$$

respectively. Here r (interest rate), μ and σ are constants and B_t is a standard Brownian motion. Througout the paper, we assume that $\mu > r$ and the two processes $\{B_t, t \geq 0\}$ and $\{\sum_{i=1}^{N_t} Y_i, t \geq 0\}$ are independent of each other. Without losing any generality, we also assume $X_0^{(0)} = 1$, and consequently $X_t^{(0)} = e^{rt}$. The other part of (3) is $X_t^{(1)} = X_0^{(1)} e^{(\mu - 0.5\sigma^2)t + \sigma B_t}$. For simplicity, we also use notation $X_t = (X_t^{(0)}, X_t^{(1)})$ from now on.

Nipp and Plum (2000) consider a similar problem without the riskless asset, or say $r = 0$. Their target is to find an investment strategy which minimizes the ruin probability. In this work we will make a change and take the the potential earning into consideration as well.

Denote the numbers of units invested in the bond and the stock at time t as $\theta_t^{(0)}, \theta_t^{(1)}$. Then, regarding to the reserve process (1), $\{\theta_t = (\theta_t^{(0)}, \theta_t^{(1)}), t \geq 0\}$ is an admissible strategy if $\{\theta_t, t \geq 0\}$ is integrable with respect to $\{X_t, t \geq 0\}$ and

$$\theta_t \cdot X_t = U + \int_0^t \theta_s \cdot dX_s + ct - \sum_{i=1}^{N_t} Y_i, \tag{4}$$

where the product $\theta_t \cdot X_t$ is the inner product of vectors. More rigorous mathematical description of admissible strategy $\{\theta_t, t \geq 0\}$ will be given in the next section. Condition (4) is actually a self-financing requirement with respect to the premium inflow ct and the claim outflow $\sum_{i=1}^{N_t} Y_i$.

Corresponding to an admissible strategy, the reserve at time t, which we still denote as R_t, is $R_t = \theta_t \cdot X_t$.

Let time 0 be the starting time and consider the expected exponential utility of the reserve at a future time T: $E(1-e^{-\alpha R_T})$. Note that R_T is possible to be negative due to the claims. So, other frequently used utility functions, like power utility function and logarithm utility function, are not suitable for the problem since they either lose the concaveness on $(-\infty, 0)$ or lose the existence on $(-\infty, 0)$. In this paper we confine the admissible strategies by requiring

$$E\int_0^T \|\theta_t \cdot X_t\|^{2^+} dt < \infty. \tag{5}$$

This requirement will save us many mathematical troubles and it does not quite affect the real applications. The problem is reduced to an optimization problem

$$\max_{\{\theta_t, 0\le t\le T\}\in\mathcal{A}} E\left[1-e^{-\alpha R_T}\right]. \tag{6}$$

Here $\mathcal{A}$ is the set of all strategies which satisfy (4) and (5).

Gerber (1970) has proposed to modify the standard form of risk reserve (1) by

$$R_t = \rho W_t + U + ct - \sum_{i=1}^{N_t} Y_i, \quad t \ge 0$$

where W_t is a standard Brownian motion and ρW_t is used to represent the uncertain environment. In this paper, we show that the optimal solution of (6) provides a R_t in a similar form

$$R_t = \frac{ve^{-r(T-t)}}{\alpha}(B_t + vt) + e^{rt}\left(U + c\int_0^t e^{-rs}ds - \sum_{i=1}^{N_t} e^{-rt_i}Y_i\right), \quad 0 \le t \le T$$

where $v = \sigma^{-1}(\mu - r)$. If there is no riskless asset (which can also be understood as the situation that the insurer invests only part of the reserve in the stock and holds the rest), we obtain by setting $r = 0$

$$R_t = \left(\frac{v}{\alpha}\right) B_t + U + \left(c + \frac{v^2}{\alpha}\right) t - \sum_{i=1}^{N_t} Y_i,$$

and it is free of the target time T. Clearly this process brings (in average) more profit than (1). With properly selected α, we will show that it is even safer than (1).

2. The martingale approach

In this paper we adopt the martingale approach as suggested in Karatzas et, al (1987) and Cox and Huang (1989) to tackle the problem. The basic idea of this approach is to transform a dynamic optimization problem into a static optimization problem.

Let $\{\mathcal{F}_t, t \geq 0\}$ be the augmented filtration generated by Brownian motion $\{B_t, t \geq 0\}$. In the studies of investment choices, it is usually assumed that the consumption process is a $\mathcal{F}_t$-adapted process. For the problem of this paper, the claim process $\{\sum_{i=1}^{N_t} Y_i, t \geq 0\}$, if viewed as a consumption process, is certainly not a $\mathcal{F}_t$-adapted process. Let $\{\mathcal{G}_t, t \geq 0\}$ be the filtration generated by claim process $\{\sum_{i=1}^{N_t} Y_i, t \geq 0\}$. We can assume that $\{\mathcal{G}_t, t \geq 0\}$ is a right-continuous filtration, because we can replace $\{\mathcal{G}_t, t \geq 0\}$ with $\{\mathcal{G}_{t^+}, t \geq 0\}$ otherwise (with respect to $\{\mathcal{G}_{t^+}, t \geq 0\}$, $\{\sum_{i=1}^{N_t} Y_i, t \geq 0\}$ still has independent increments due to condition (2) and the fact that each path of $\{\sum_{i=1}^{N_t} Y_i, t \geq 0\}$ is right-continuous). Let $(\Omega_1, \mathcal{F}_T(\mathcal{F}_t), P_1)$ and $(\Omega_2, \mathcal{G}_T(\mathcal{G}_t), P_2)$ be the two complete propability spaces containing process $\{B_t, 0 \leq t \leq T\}$ and the claim process $\{\sum_{i=1}^{N_t} Y_i, 0 \leq t \leq T\}$ respectively. Then the probability space we will be working on is the product space $(\Omega = \Omega_1 \otimes \Omega_2, \mathcal{F}_T \otimes \mathcal{G}_T(\mathcal{F}_t \otimes \mathcal{G}_t), P = P_1 \otimes P_2)$. The asset price process $\{X_t, 0 \leq t \leq T\}$, by definition (3), is a semimartingale in $(\Omega_1, \mathcal{F}_T(\mathcal{F}_t), P_1)$, which certainly can be viewed as a semimartingale in product space $(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T(\mathcal{F}_t \otimes \mathcal{G}_t), P)$ as well. A strategy $\{\theta_t, 0 \leq t \leq T\}$, in mathematical terminology, is an $\mathcal{F}_t \otimes \mathcal{G}_t$-adapted process integrable with respect to the semimartingale $\{X_t, 0 \leq t \leq T\}$.

Denote $\hat{X}_t = e^{-rt} X_t$ and $\hat{R}_t = e^{-rt} R_t$. Then, we have

Lemma. For any $\{\theta_t, 0 \leq t \leq T\} \in \mathcal{A}$, condition (4) is equivalent to

$$\hat{R}_t = \theta_t \cdot \hat{X}_t = U + \int_0^t \theta_s \cdot d\hat{X}_s + c \int_0^t e^{-rs} ds - \sum_{i=1}^{N_t} e^{-rt_i} Y_i \tag{7}$$

where t_i is the time point when the i-th claim arrives.

This equivalence is well known if all the processes got involved are continous processes (see Chapter 9 of Duffie (1996)). Although the claim process here is a process with jumps, it does not change the equivalence. For completeness, we write out the proof below.

Proof: We present only the derivation of (7) from (4). One can easily get the other half of the proof by reversing the derivation.

Denote

$$I_t = \int_0^t \theta_s \cdot dX_s, \quad J_t = U + ct - \sum_{i=1}^{N_t} Y_i.$$

Then $\{I_t, 0 \leq t \leq T\}$ is a continous semimartingale (in $(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P)$) and $\{J_t, 0 \leq t \leq T\}$ is a process with finite variation with probability one. Since $\theta_t \cdot \hat{X}_t = e^{-rt} (\theta_t \cdot X_t)$, we have from (4)

$$\theta_t \cdot \hat{X}_t - \theta_0 \cdot \hat{X}_0 = e^{-rt} I_t - e^{-r0} I_0 + e^{-rt} J_t - e^{-r0} J_0.$$

With integration by parts formula (see page 155 of Karatzas and Shreve (1991) and note that the cross-variation term is zero in this case), we have

$$\begin{aligned} e^{-rt} I_t - e^{-r0} I_0 &= \int_0^t e^{-rs} dI_s + \int_0^t I_s de^{-rs} \\ &= \int_0^t e^{-rs} \theta_s \cdot dX_s + \int_0^t I_s de^{-rs}, \end{aligned}$$

and

$$\begin{aligned} e^{-rt} J_t - e^{-r0} J_0 &= \int_0^t e^{-rs} dJ_s + \int_0^t J_s de^{-rs} \\ &= c \int_0^t e^{-rs} ds - \sum_{i=1}^{N_t} e^{-rt_i} Y_i + \int_0^t J_s de^{-rs}. \end{aligned}$$

Here the integral $\int_0^t e^{-rs} dJ_s$ is simply the Riemann-Stieltjes integral along the paths (since almost surely each path of $\{J_t, 0 \le t \le T\}$ has finite variation). From (4), $I_s + J_s = \theta_s \cdot X_s$. Thus, summing up the above two equalities, we get

$$\theta_t \cdot \hat{X}_t - \theta_0 \cdot \hat{X}_0 = \int_0^t e^{-rs} \theta_s \cdot dX_s + \int_0^t \theta_s \cdot X_s de^{-rs} + c \int_0^t e^{-rs} ds - \sum_{i=1}^{N_t} e^{-rt_i} Y_i.$$

Again by the integration by parts formula, we have

$$\int_0^t e^{-rs} \theta_s \cdot dX_s + \int_0^t \theta_s \cdot X_s de^{-rs} = \int_0^t \theta_s \cdot \left(e^{-rs} dX_s + X_s de^{-rs} \right) = \int_0^t \theta_s \cdot d\hat{X}_s.$$

The equivalence (7) thus follows by noting that $\theta_0 \cdot \hat{X}_0 = U$. $\diamondsuit$

Now we set to introduce a static program.

Define

$$\xi_t := \exp\left(-vB_t - \frac{v^2 t}{2} \right) \tag{8}$$

where $v = \sigma^{-1}(\mu - r)$, and define a probability measure Q_1 on $(\Omega_1, \mathcal{F}_T)$ as

$$Q_1(A) = E_1[1_A \xi_T], \ \forall A \in \mathcal{F}_T \ (E_1 \text{ is with respect to } P_1). \tag{9}$$

Confined on $\mathcal{F}_t$ for any $0 \le t \le T$, the Radon-Nikodym derivative dQ_1/dP_1 is ξ_t. By Girsanov's theorem (see page 191 of Karatzas and Shreve (1991)), $\{\hat{B}_t = B_t + vt, 0 \le t \le T\}$ is a standard Brownian motion in probability space $(\Omega_1, \mathcal{F}_T(\mathcal{F}_t), Q_1)$ and the discounted stock index $\{\hat{X}_t^{(1)}, 0 \le t \le T\}$ is a martingale in $(\Omega_1, \mathcal{F}_T(\mathcal{F}_t), Q_1)$ with dynamics

$$d\hat{X}_t^{(1)} = \sigma \hat{X}_t^{(1)} d\hat{B}_t.$$

In probability space $(\Omega = \Omega_1 \otimes \Omega_2, \mathcal{F}_T \otimes \mathcal{G}_T(\mathcal{F}_t \otimes \mathcal{G}_t), Q = Q_1 \otimes P_2)$, $\{\hat{B}_t, 0 \le t \le T\}$ is still a standard Brownian motion and $\{\hat{X}_t^{(1)}, 0 \le t \le T\}$ is still a martingale (with respect to filtration $\{\mathcal{F}_t \otimes \mathcal{G}_t, 0 \le t \le T\}$). Note that $d\hat{X}_t^{(0)} = 0$ and, from condition (5) and the fact that ξ_T has moments of any order under P, one can prove by Holder's inequality that

$$E^Q \left(\int_0^T \left\| \theta_t \cdot \hat{X}_t \right\|^2 dt \right) < \infty.$$

Here E^Q is the expectation under measure Q (expectation under physical measure P is simply written as E). So, according to the theory of Ito's integration, $\{\int_0^t \theta_s \cdot d\hat{X}_s, 0 \le t \le T\}$ is a zero mean martingale in $(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T(\mathcal{F}_t \otimes \mathcal{G}_t), Q)$, and hence $E^Q(\int_0^T \theta_t \cdot d\hat{X}_t) = 0$, which, expressed under probability measure P, is

$$\begin{aligned} E^Q \left(\int_0^T \theta_t \cdot d\hat{X}_t \right) &= E \left[\xi_T \left(\int_0^T \theta_t \cdot d\hat{X}_t \right) \right] \\ \text{(by (7))} &= E \left[\xi_T \left(e^{-rT} R_T - \int_0^T c e^{-rt} dt + \sum_{i=1}^{N_T} e^{-rt_i} Y_i - U \right) \right] \\ &= 0. \end{aligned}$$

Recall the independence of the stock index and the claim process and note that $E\xi_t = 1$ for any t. We have

$$E \left(e^{-rT} R_T \xi_T + \sum_{i=1}^{N_T} e^{-rt_i} Y_i \right) = U + c \int_0^T e^{-rt} dt \tag{10}$$

for R_T corresponding to any strategy in $\mathcal{A}$. Also, under the condition (5), $\int_0^t \theta_s \cdot dX_s$ is square-integrable. Together with condition (2), we have thus from (4)

$$E(R_T^2) < \infty. \tag{11}$$

Write (11) and (10) as constraints:

$$\begin{aligned} &V \in L^2(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P) \\ &E \left(e^{-rT} V \xi_T + \sum_{i=1}^{N_T} e^{-rt_i} Y_i \right) = U + c \int_0^T e^{-rt} dt. \end{aligned}$$

A static program is then introduced as

$$\begin{cases} \max \quad E\left(1-e^{-\alpha V}\right) \\ s.t. \quad V \in L^2(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P) \\ \qquad\quad E\left(e^{-rT}V\xi_T + \sum_{i=1}^{N_T} e^{-rt_i}Y_i\right) = U + c\int_0^T e^{-rt}dt. \end{cases} \tag{12}$$

This optimization problem is not equivalent with the original problem (6), but it is helpful for the solving of (6).

Program (12) can be viewed as a functional optimization problem in Hilbert space $L^2(\Omega, \mathcal{F}_T\otimes\mathcal{G}_T, P)$ and be solved by Lagrange multiplier method (see for example Chapter 8 of Luenberger (1969)).

Proposition 1. The solution of program (12) is

$$V^* = \frac{vB_T + v^2T}{\alpha} + e^{rT}\left[U + c\int_0^T e^{-rt}dt - E\left(\sum_{i=1}^{N_T} e^{-rt_i}Y_i\right)\right]. \tag{13}$$

Proof: Program (12) is the same as

$$\begin{cases} \min \quad E\left(e^{-\alpha V}\right) \\ s.t. \quad V \in L^2(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P) \\ \qquad\quad E\left(e^{-rT}V\xi_T + \sum_{i=1}^{N_T} e^{-rt_i}Y_i\right) = U + c\int_0^T e^{-rt}dt, \end{cases}$$

which is a constrained convex program on Hilbert space $L^2(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P)$. It is easy to check that the optimal value $\min E\left(e^{-\alpha V}\right)$ is finite by choosing a special V (constant):

$$V = e^{rT}\left[U + c\int_0^T e^{-rt}dt - E\left(\sum_{i=1}^{N_T} e^{-rt_i}Y_i\right)\right]$$

which clearly satisfies the constraints. Let

$$H(V) = E\left(e^{-\alpha V}\right) + \lambda\left[E\left(e^{-rT}\xi_T V + \sum_{i=1}^{N_T} e^{-rt_i}Y_i\right) - c\int_0^T e^{-rt}dt - U\right].$$

Then, by the theory of convex program (see page 216-218 and Problem 7 of page 236 of Luenberger (1969)), the optimal value must be achieved at a V such that for any (unit norm) $h \in L^2(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P)$

$$\delta H(V,h) = 0, \quad \text{and} \quad \lambda\left[E\left(e^{-rT}\xi_T V + \sum_{i=1}^{N_T} e^{-rt_i}Y_i\right) - c\int_0^T e^{-rt}dt - U\right] = 0.$$

Here $\delta H(V,h)$ is the Gateaux derivative of functional H (at V) along direction h. It is easy to verify that

$$\delta H(V,h) = E\left[\left(\lambda e^{-rT}\xi_T - \alpha e^{-\alpha V}\right)h\right].$$

So, in order to have $\delta H(V,h)=0$ for any h, the term inside the bracket must be zero, i.e., $\alpha e^{-\alpha V}=\lambda e^{-rT}\xi_T$, and hence

$$\alpha V = rT + vB_T + \frac{v^2T}{2} - \log\left(\frac{\lambda}{\alpha}\right).$$

From this equation, together with

$$E\left(e^{-rT}V\xi_T + \sum_{i=1}^{N_T} e^{-rt_i}Y_i\right) - U - c\int_0^T e^{-rt}dt = 0,$$

one can obtain

$$V = \frac{vB_T + v^2T}{\alpha} + e^{rT}\left[U + c\int_0^T e^{-rt}dt - E\left(\sum_{i=1}^{N_T} e^{-rt_i}Y_i\right)\right]$$

by noting at the facts $E\xi_T = 1$, $E(vB_T \cdot e^{-vB_T}) = -v^2T \cdot e^{v^2T/2}$. $\diamondsuit$

3. The optimal R_T and the optimal strategy

The feasible set of program (12) is larger than the set of outcomes corresponding to strategies in $\mathcal{A}$. This means that, for a $V \in \mathcal{F}_T \otimes \mathcal{G}_T$ which is feasible to program (12), there may not exist a admissible strategy in $\mathcal{A}$ such that the outcome R_T of this strategy is V. The optimal solution V^* is unfortunately one which is not attainable via an admissible strategy. The reason is as the following.

If a strategy $\{\theta_t^*, 0 \le t \le T\} \in \mathcal{A}$ exists and leads to an outcome R_T which is equal to V^*, then $\{\int_0^t \theta_s^* \cdot d\hat{X}_s, 0 \le t \le T\}$ is a martingale under Q and, according to (13) and (7),

$$\begin{aligned}
\int_0^t \theta_s^* \cdot d\hat{X}_s &= E^Q\left[\int_0^T \theta_s^* \cdot d\hat{X}_s \middle| \mathcal{F}_t \otimes \mathcal{G}_t\right] \\
&= E^Q\left[\hat{V}^* - U - c\int_0^T e^{-rs}ds + \sum_{i=1}^{N_T} e^{-rt_i}Y_i \middle| \mathcal{F}_t \otimes \mathcal{G}_t\right] \\
&= E^Q\left[\frac{ve^{-rT}\hat{B}_T}{\alpha} - E\left(\sum_{i=1}^{N_T} e^{-rt_i}Y_i\right) + \sum_{i=1}^{N_T} e^{-rt_i}Y_i \middle| \mathcal{F}_t \otimes \mathcal{G}_t\right]
\end{aligned}$$

where $\hat{V}^* = V^*e^{-rT}$. Note that the probability law of the claim process is the same under both measure Q and measure P, and the claim process has independent increments. So

$$\begin{aligned}
\int_0^t \theta_s^* \cdot d\hat{X}_s &= \frac{ve^{-rT}}{\alpha}E^Q\left[\hat{B}_T \middle| \mathcal{F}_t \otimes \mathcal{G}_t\right] + \sum_{i=1}^{N_t} e^{-rt_i}Y_i - E\left(\sum_{i=1}^{N_t} e^{-rt_i}Y_i\right) \\
&= \frac{ve^{-rT}}{\alpha}\hat{B}_t + \sum_{i=1}^{N_t} e^{-rt_i}Y_i - E\left(\sum_{i=1}^{N_t} e^{-rt_i}Y_i\right).
\end{aligned}$$

Taking into account $d\hat{X}_t^{(0)} = 0$, $d\hat{X}_t^{(1)} = \sigma \hat{X}_t^{(1)} d\hat{B}_t$, we further have

$$\int_0^t \left(\sigma \theta_s^{*(1)} - \frac{ve^{-rT}}{\alpha \sigma \hat{X}_s^{(1)}} \right) d\hat{X}_s^{(1)} = \sum_{i=1}^{N_t} e^{-rt_i} Y_i - E\left(\sum_{i=1}^{N_t} e^{-rt_i} Y_i \right). \tag{14}$$

Since $\{\theta_t^*, 0 \le t \le T\}$ satisfies requirement (5), the left side of (14) is a continuous process (it is a continuous semimartingale in $(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, P)$ and a continuous martingale in $(\Omega, \mathcal{F}_T \otimes \mathcal{G}_T, Q)$), while the right side is a martingale with jumps. The equality therefore can not hold, and hence the existence of strategy $\{\theta_t^*, 0 \le t \le T\}$ is not true.

Although (13) is unattainable, it provides a clue and an attainable outcome which is almost the same as (13) is

$$V^+ = \frac{vB_T + v^2 T}{\alpha} + e^{rT} \left[U + c \int_0^T e^{-rt} dt - \sum_{i=1}^{N_T} e^{-rt_i} Y_i \right]. \tag{15}$$

The evidence is intuitively as the following. Let us temporarily forget about the claim process and consider an investment problem with initial fund U and continuous input rate c. With the same procedure as in the proof for Proposition 1 one can see that the optimal outcome corresponding to the exponential utility function $1 - e^{-\alpha x}$ is

$$\frac{vB_T + v^2 T}{\alpha} + e^{rT} \left[U + c \int_0^T e^{-rt} dt \right].$$

Now bring back the claim process. If the insurer pays the claimer by selling (or short-selling) the riskless asset, the bond, whenever a claim arrives, then the final result at time T is exactly of form (15). In fact, we have

Proposition 2. V^+ of (15) is the optimal reserve (at time T) determined by optimization problem (6).

We present the strategy to realise V^+ first and present the proof of Proposition 2 later.

Proposition 3. Let $\{\theta_t^+, 0 \le t \le T\}$ be the strategy which leads to the outcome V^+ at time T. Then, the amount put into the stock at time t $(0 \le t \le T)$ is

$$\theta_t^{+(1)} X_t^{(1)} = \frac{ve^{-r(T-t)}}{\alpha \sigma} \tag{16}$$

and the total reserve at time t is

$$R_t^+ = \frac{ve^{-r(T-t)}}{\alpha} (B_t + vt) + e^{rt} \left(U + c \int_0^t e^{-rs} ds - \sum_{i=1}^{N_t} e^{-rt_i} Y_i \right). \tag{17}$$

The amount in the bond is $\theta_t^{+(0)} X_t^{(0)} = R_t^+ - \theta_t^{+(1)} X_t^{(1)}$.

Proof: By (15) and (7),

$$\int_0^t \theta_s^+ \cdot d\hat{X}_s = E^Q\left[\hat{V}^+ - U - \int_0^T e^{-rs} f(s)ds + \sum_{i=1}^{N_T} e^{-rt_i} Y_i \middle| \mathcal{F}_t \otimes \mathcal{G}_t\right] = \frac{ve^{-rT}}{\alpha}\hat{B}_t \quad (18)$$

where $\hat{V}^+ = V^+ e^{-rT}$. Comparing the coefficients of both sides and bearing in mind the facts $d\hat{X}_t^{(0)} = 0$, $d\hat{X}_t^{(1)} = \sigma \hat{X}_t^{(1)} d\hat{B}_t$, we see that the units in the stock $\theta_t^{+(1)}$ follows

$$\sigma \hat{X}_t^{(1)} \theta_t^{+(1)} = \frac{ve^{-rT}}{\alpha}, \quad t \geq 0$$

or,

$$\theta_t^{+(1)} = \frac{ve^{-rT}}{\alpha\sigma\hat{X}_t^{(1)}} = \frac{ve^{-r(T-t)}}{\alpha\sigma X_t^{(1)}}, \quad t \geq 0.$$

Bringing (18) back to (7), we get

$$\hat{R}_t^+ = \theta^+ \cdot \hat{X}_t = U + \frac{ve^{-rT}}{\alpha}\hat{B}_t + c\int_0^t e^{-rs}ds - \sum_{i=1}^{N_t} e^{-rt_i} Y_i$$

or,

$$R_t^+ = \theta^+ \cdot X_t = \frac{ve^{-r(T-t)}}{\alpha}(B_t + vt) + e^{rt}\left(U + c\int_0^t e^{-rs}ds - \sum_{i=1}^{N_t} e^{-rt_i} Y_i\right).$$

The units in the bond $\theta_t^{+(0)}$ is easy to write out from

$$\theta_t^{+(0)} X_t^{(0)} = R_t^+ - \theta_t^{+(1)} X_t^{(1)}.$$

It is easy to check that strategy $\{\theta_t^+, 0 \leq t \leq T\}$ satisfies (5) directly from the above expression of $\theta^+ \cdot X_t$. $\diamondsuit$

Note that $X_t^{(1)} = X_0^{(1)} e^{(\mu - 0.5\sigma^2)t + \sigma B_t}$. Thus, (17) can be changed into an expression in terms of the price process $X_t^{(1)}$.

Proof of Proposition 2: Since V^+ is attainable, what we need to show for the rest is

$$E\left(1 - e^{-\alpha R_T}\right) \leq E\left(1 - e^{-\alpha V^+}\right)$$

for any R_T corresponding to a strategy in $\mathcal{A}$.

Consider conditional expectation $E[\,\cdot \mid N_T, (t_i, Y_i)_{i \leq N_T}]$. With the same procedure as in the proof of Proposition 1, we can prove that, for given $N_T, (t_i, Y_i)_{i \leq N_T}$, V^+ of (15) is the solution of program

$$\begin{cases} \max & E_1\left(1 - e^{-\alpha V}\right) \\ s.t. & V \in L^2(\Omega, \mathcal{F}_T, P_1) \\ & E_1\left(e^{-rT} V \xi_T\right) = U + c\int_0^T e^{-rt}dt - \sum_{i=1}^{N_T} e^{-rt_i} Y_i. \end{cases}$$

On the other hand, for the given $N_T, (t_i, Y_i)_{i \le N_T}$, the condition (10) for R_T (corresponding to a stategy in $\mathcal{A}$) becomes

$$E_1\left(e^{-rT} R_T \xi_T\right) = U + c\int_0^T e^{-rt}dt - \sum_{i=1}^{N_T} e^{-rt_i} Y_i$$

and requirement (5) leads to $E_1(R_T)^2 < \infty$. We conclude therefore

$$E\left[1 - e^{-\alpha R_T} \middle| N_T, (t_i, Y_i)_{i \le N_T}\right] \le E\left[1 - e^{-\alpha V^+} \middle| N_T, (t_i, Y_i)_{i \le N_T}\right].$$

for a R_T such that $E\left(1 - e^{-\alpha R_T}\right)$ is finite. And hence

$$\begin{aligned} E\left(1 - e^{-\alpha R_T}\right) &= E\left\{E\left[1 - e^{-\alpha R_T} \middle| N_T, (t_i, Y_i)_{i \le N_T}\right]\right\} \\ &\le E\left\{E\left[1 - e^{-\alpha V^+} \middle| N_T, (t_i, Y_i)_{i \le N_T}\right]\right\} \\ &= E\left(1 - e^{-\alpha V^+}\right). \end{aligned}$$

If $E(1-e^{-\alpha R_T})$ is infinite, it must be $-\infty$. So we still have $E(1-e^{-\alpha R_T}) < E(1-e^{-\alpha V^+})$. The proof is thus completed. $\diamondsuit$

4. Improvements resulting from the investment

For the purpose of comparing with the classical model (1), we consider the case $r = 0$. As mentioned in the introduction, this can be understood as the situation without riskless asset, or, as in Nipp and Plum (2000), a preference of investing only part of the reserve in risky asset and holding the rest.

Setting $r = 0$ in (17) gives

$$R_t^+ = \left(\frac{v}{\alpha}\right) B_t + U + \left(c + \frac{v^2}{\alpha}\right) t - \sum_{i=1}^{N_t} Y_i. \tag{19}$$

The expression is free of the target time T! In fact, the amount of the reserve invested into the risk asset (stock) is fixed regardless of the target time T. In the following, we consider the risk associated with process R_t^+ with no restriction on the time horizon. We will be satisfied with the comparison of adjustment coefficients with respect to (19) and (1) since few explicit results are available for the exact ruin probability of the model.

Suppose $Y_i, i \ge 1$ are i.i.d. random variables whose moment generating function $M(\gamma) = E(e^{\gamma Y_1})$ exists in $[0, a)$ for some positive a. Suppose also $\{N_t, t \ge 0\}$ is a Poisson process (independent of the claim amounts) with arrive rate β and $c > \beta E(Y_1)$ (positive loading). Corresponding to R_t^+ of (19), define the time of ruin as

$$\tau = \inf\{t \ge 0 : R_t^+ < 0\} \quad (\inf\{\emptyset\} := \infty).$$

By this definition, τ is an optional time of filtration $\{\mathcal{F}_t \otimes \mathcal{G}_t, t \geq 0\}$. It is also a stopping time of filtration $\{\mathcal{F}_t \otimes \mathcal{G}_t, t \geq 0\}$ since the filtration is right-continuous. A adjustment coefficient (with respect to (19)) is a positive constant γ such that $\{e^{\gamma(U-R_t^+)}, t \geq 0\}$ is a martingale. If such a γ exists, applying optional sampling theorem to $E[e^{\gamma(U-R^+_{\tau\wedge T})}]$ and then setting $T \to \infty$ (see Proposition 1.1 of Asmussen 2000 and check the requirement there by the iterated logarithm law of Brownian motion and the fact that $\{\sum_{i=1}^n [c(t_i - t_{i-1}) - Y_i], n \geq 1\}$ is a random walk with positive drift), we get

$$P\left(\min_{0\leq t<\infty} R_t^+ < 0\right) = P(\tau < \infty) < e^{-\gamma U},$$

which is generally called Lundberg inequality.

Now we solve γ and make comparison with the adjustment coefficient of model (1). Clearly,

$$\begin{aligned}
&E\left[e^{\gamma(U-R^+_{t+s})}\middle| \mathcal{F}_s \otimes \mathcal{G}_s\right] \\
&= e^{\gamma(U-R_s^+)} E\left\{\exp\left[\gamma \sum_{i=N_s+1}^{N_{t+s}} Y_i - \gamma\left(c + \frac{v^2}{\alpha}\right) t - \gamma\left(\frac{v}{\alpha}\right)(B_{t+s} - B_s)\right]\right\} \\
&= e^{\gamma(U-R_s^+)} E\left\{\exp\left[\gamma \sum_{i=1}^{N_t} Y_i - \gamma\left(c + \frac{v^2}{\alpha}\right) t - \gamma\left(\frac{v}{\alpha}\right) B_t\right]\right\}
\end{aligned}$$

since $\{R_t^+, t \geq 0\}$ has independent increments and $\{N_t, t \geq 0\}$ is a Possion process. A simple calculation gives

$$E\exp\left\{\gamma\left[\sum_{i=1}^{N_t} Y_i - \left(c + \frac{v^2}{\alpha}\right) t - \left(\frac{v}{\alpha}\right) B_t\right]\right\} = \exp\left[\beta t(M(\gamma) - 1) - \gamma\left(c + \frac{v^2}{\alpha}\right) t + \frac{\gamma^2 v^2 t}{2\alpha^2}\right].$$

The adjustment coefficient is thus the solution of equation

$$\beta\left[M(\gamma) - 1\right] - \gamma\left(c + \frac{v^2}{\alpha}\right) + \frac{\gamma^2 v^2}{2\alpha^2} = 0. \tag{20}$$

The adjustment coefficient for the classical collective model (1) is well known to be the solution of the equation

$$\beta(M(\gamma) - 1) - c\gamma = 0. \tag{21}$$

The left side of equation (21) and the left side of equation (20) are both convex about γ and, with the positive loading requirement, they go down first from 0 to negative values and then go up to positive infinity. So, there must be solutions. To distinguish, denote γ_0 and γ_1 as the solution of (21) and (20). The Lundberg inequalities of model (1) and (19) are then

$$P\left(\min_{0\leq t<\infty} R_t < 0\right) < e^{-\gamma_0 U}, \quad P\left(\min_{0\leq t<\infty} R_t^+ < 0\right) < e^{-\gamma_1 U}.$$

Comparing the two equations, one can easily see that

$$\gamma_1 \geq \gamma_0, \quad \text{if } 2\alpha \geq \gamma_0. \tag{22}$$

This shows that a properly selected α will increase the degree of safety of the insurer.

For whatever α, the average earning speed of (14), $c + \alpha^{-1}v^2 - \beta E(Y_1)$, is always greater than $c - \beta E(Y_1)$, the average earning speed of (1). For the best earning speed without compromising on safety, the choice of α should be $\gamma_0/2$.

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